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(May–June)

FYUP : 4th Semester Examination

MATHEMATICS

(Matrix Theory and Vector Spaces)

(MTH–253)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **four** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Prove that every square matrix can be expressed uniquely as a sum of a symmetric matrix and a skew-symmetric matrix. 5

(b) Find the conjugate transpose of the matrix

$$\begin{bmatrix} 1+i & -3+2i & 7-i \\ 2+5i & 9 & 8-3i \\ -7i & 4+i & -6+7i \end{bmatrix}$$ 3

(c) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$$

using elementary operations. 5

(d) Let A and B be square matrices of the same order. Show that

(i) $A + A^{\theta}$ is Hermitian

(ii) $A - A^{\theta}$ is skew-Hermitian

where A^{θ} is the conjugate transpose of the matrix A. 2½+2½=5

2. (a) If A and B are two n-rowed non-singular matrices, then show that AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$. 1+4=5

(b) Show that

(i) $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent;

(ii) $\begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$ is idempotent. 2½+1½=4

(c) Reduce the following matrix to echelon form and hence find its rank : 4+1=5

$$\begin{bmatrix} 3 & -5 & -1 \\ 4 & -6 & 0 \\ 1 & -2 & -1 \end{bmatrix}$$

(d) If A and B are two matrices conformable for multiplication, then show that $(AB)^T = B^T A^T$. 4

UNIT—II

3. (a) If A is any n-rowed square matrix, then prove that $|\text{Adj } A| = |A|^{n-1}$. 5

(b) Solve the following system of linear equations using Cramer's rule : 5

$$x + y + 4z = 6$$
$$3x + 2y - 2z = 9$$
$$5x + y + 2z = 13$$

(c) Determine the solutions of the following system of homogeneous equations : 5

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

(d) Find the inverse of the following matrix using adjoint of the matrix : 3

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

(e) What is determinantal rank of a matrix? 1

4. (a) Show that the following system of equations is consistent and hence determine its solutions : 3+3=6

$$x+y+z=9$$
$$2x+5y+7z=52$$
$$2x+y-z=0$$

(b) If A be any n-rowed square matrix, then prove that

$$(\text{Adj}\, A)A = A(\text{Adj}\, A) = |A| I_n$$ 5

(c) Show that the following system of homogeneous equations has trivial solution only : 4

$$x+2y+3z=0$$
$$3x+4y+4z=0$$
$$7x+10y+12z=0$$

(d) For what value(s) of the parameter λ will the system of equations fail to have a unique solution? 4

$$3x-y+\lambda z=1$$
$$2x+y+z=2$$
$$x+2y-\lambda z=-1$$

UNIT—III

5. (a) Show that $\mathbb{R}^2$ is not a vector space over $\mathbb{R}$ with respect to addition and scalar multiplication defined as follows : 4

For (u_1, u_2), $(v_1, v_2) \in \mathbb{R}^2$, $a \in \mathbb{R}$

$$(u_1, u_2)+(v_1, v_2)=(u_1+v_1, u_2+v_2)$$
$$a(u_1, u_2)=(au_1, u_2)$$

(b) Prove that the intersection of two subspaces of a vector space is also a subspace. 4

(c) If $V(F)$ is a finite dimensional vector space and W a subspace of V, then prove that the quotient space V/W is also finite dimensional, and

$$\dim V/W = \dim V - \dim W$$ 1+5=6

(d) Show that the set $S=\{x-1, x^2-1\}$ forms a basis of the vector space

$$W=\{a_0+a_1x+a_2x^2 \in \mathbb{R}[x] \mid a_0+a_1+a_2=0\}$$

over the field $\mathbb{R}$. 5

6. (a) If V is a vector space over a field F, and W_1, W_2 are two subspaces of V, then prove that $W_1 + W_2$ is also a subspace of V, containing W_1 and W_2. Moreover, prove that it is the smallest subspace containing W_1 and W_2. 2+3=5

(b) Let S be a non-empty set. Show that the set V of all functions from S to $\mathbb{R}$ is a vector space over $\mathbb{R}$, with respect to addition and scalar multiplication defined below : 6

$$(f+g)(x) = f(x) + g(x),\ \forall x \in S,\ f,\ g \in V$$
$$(af)(x) = af(x),\ \forall x \in S,\ a \in \mathbb{R}$$

(c) In the vector space $\mathbb{R}^3(\mathbb{R})$, show that the vectors $v_1 = (1, 2, 3)$, $v_2 = (0, 1, 2)$ and $v_3 = (2, 3, 1)$ are linearly independent. 3

(d) If $V(F)$ is a finite dimensional vector space with $\dim V = n$ and W is any subspace of $V(F)$, then prove that $\dim W \le \dim V$. 5

UNIT—IV

7. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x+z, y+z, y+z)$$

(i) Show that T is a linear transformation.

(ii) Find the matrix of T with respect to the standard basis of $\mathbb{R}^3$. 2+4=6

(b) Let U and V be vector spaces over a field F and $T : U \to V$ be a linear transformation from U onto V and $W = \ker T$. Then prove that $U / W \cong V$. 5

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$$T(x, y, z) = (3x, x-y, 2x+y+z)$$

Show that T is invertible and find the rule by which T^{-1} is defined. 4+1=5

(d) Find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$ 3

8. (a) Find the range, rank, kernel and nullity of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x, y) = (x, x+y, z)$$ 3+1+3+1=8

(b) State Cayley-Hamilton theorem. 2

(c) Let $T : U \to V$ be a linear transformation, where U and V are finite dimensional vector spaces over the same field F. Prove that

$$\operatorname{rank} T + \operatorname{nullity} T = \dim U$$ 5

(d) Define the following : 2+1+1=4

(i) Eigenvalues and eigenvectors of a matrix

(ii) Minimal polynomial of a linear operator

(iii) Monic polynomial of a linear operator

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